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(a) Contract AF04(695)-113, Exhibit "A"
(b) AFBM Exhibit 58-1, Paragraph 3.18
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Western Development Laboratories

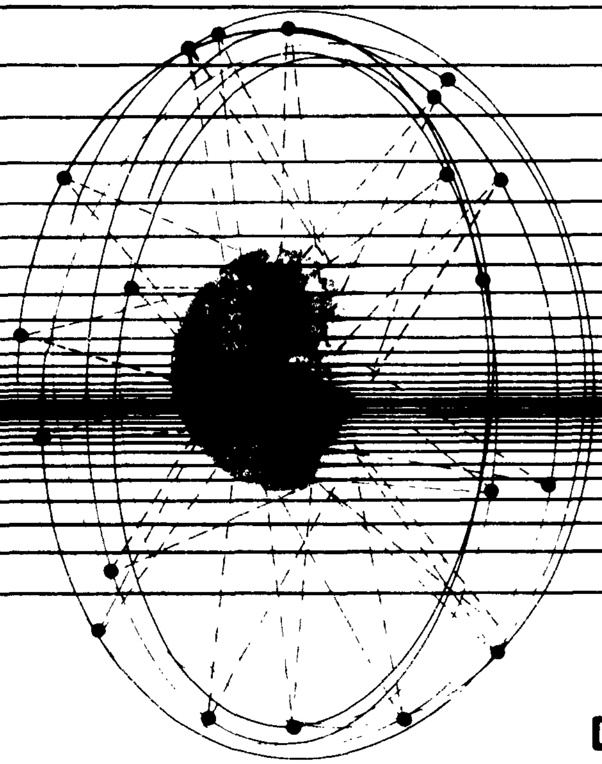
R W Boyd

R. W. Boyd
Manager, Contracts Management

TECHNICAL OPERATING REPORT

WDL-TR1900

25 JANUARY 1963



# DETERMINATION OF THE SPECTRUM OF A CARRIER PHASE MODULATED BY A BI-PHASE SUBCARRIER



PREPARED FOR

AIR FORCE SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
INGLEWOOD, CALIFORNIA

CONTRACT AF04(695)-113

PHILCO
A SUBSIDIARY OF Ford Motor Company

WESTERN DEVELOPMENT LABORATORIES
PALO ALTO, CALIFORNIA

WDL-TR1900
25 January 1963

TECHNICAL OPERATING REPORT

# DETERMINATION OF THE SPECTRUM OF A CARRIER PHASE-MODULATED BY A BI-PHASE SUBCARRIER

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Contract AF04(695)-113
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AFSSD Exhibit 61-27A, Para. 1.2.1.1

Prepared for

SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
Inglewood, California

## ABSTRACT

PHILCO WDL-TR1900 UNCLASSIFIED
DETERMINATION OF THE SPECTRUM OF A CARRIER PHASE-MODULATED BY A BI-PHASE SUBCARRIER
18 pages
25 January 1963
Contract AF04(695)-113

This Technical Operating Report presents a method for determining the spectrum of a carrier that is phase-modulated by a phase subcarrier. Derivations of the power spectrum of the phase subcarrier and the relative magnitude of the phase modulator output are included.

## FOREWORD

This Technical Operating Report on Definitive Contract AF04(695)-113 is submitted in accordance with Exhibit "A" of that contract and Paragraph 3.18 of AFBM Exhibit 58-1, "Contractor Reports Exhibit," dated 1 October 1959, as revised and amended.

This report was prepared by Philco Western Development Laboratories in fulfilling the requirements of Paragraph 1.2.1.1 of AFSSD Exhibit 61-27A, Satellite Control Subsystem Work Statement, dated 15 February 1962, as revised and amended.

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## APPENDIX

## SECTION 1

## INTRODUCTION

It is desired to obtain the spectrum of a bi-phase subcarrier, that switches phase every bit period, phase-modulating a carrier. Much information on system requirements can be obtained from this spectrum.

The condition described above implies the worst case with regard to bandwidth. As will be seen, the minimum i-f bandwidth is twice the sum of the subcarrier frequency and one-half the switching frequency. It will also be seen that components beyond the third harmonic are appreciably small and may be neglected.

The means of mechanizing the above system, important waveforms, the power spectrum of the bi-phase subcarrier, and the relative magnitudes of the phase modulator output are shown in Figs. 1 through 4. The derivation of the latter two figures is given in Section 2.

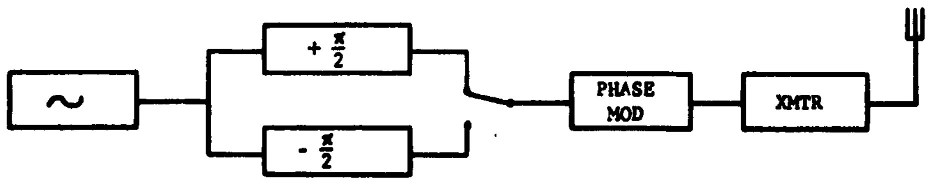



Fig. 1
Mechanization of Biphase Modulated Carrier

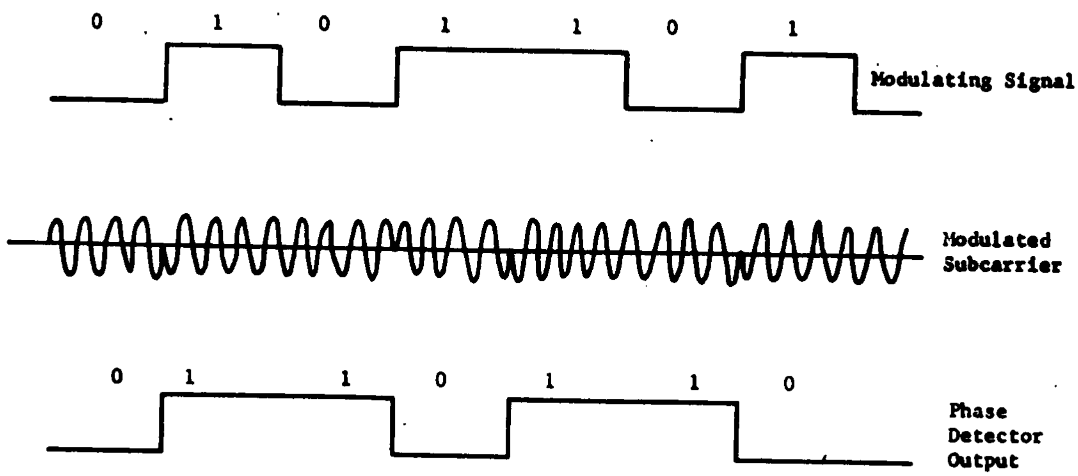
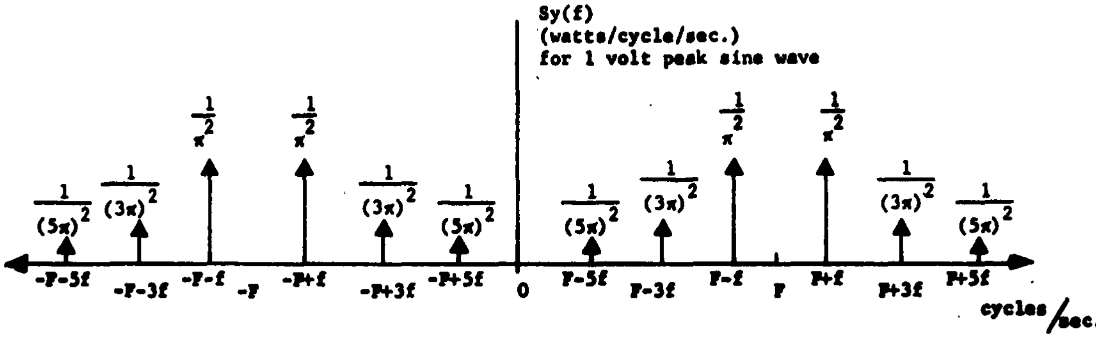



Fig. 2
Important System Waveforms



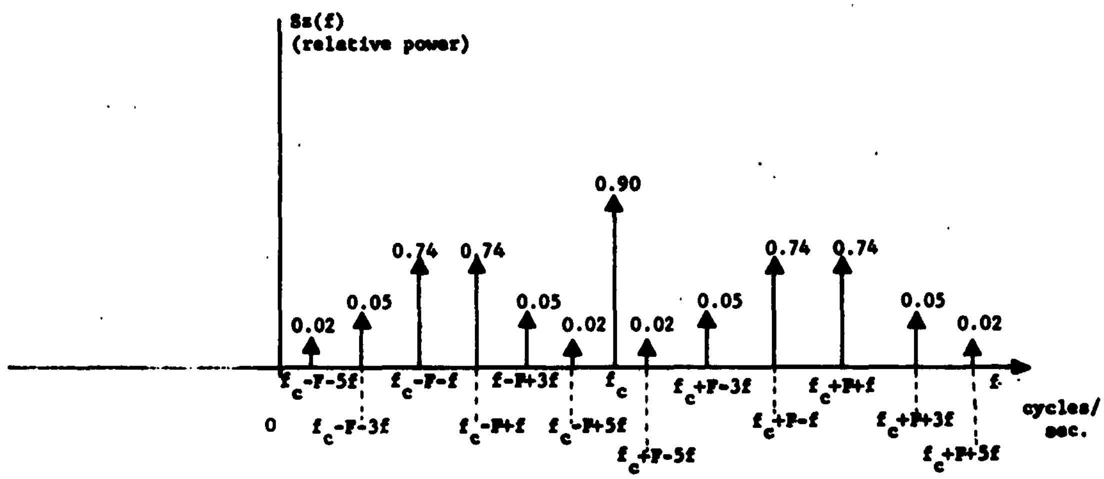

$f = \frac{1}{2T}$   T = bit period

2T = wave period

F = subcarrier frequency

Fig. 3

Power Spectrum of Biphase Subcarrier




$f_c$ = carrier frequency

F = subcarrier frequency

f = 1/2 switching rate

Fig. 4

Relative Power Spectrum of Modulator Output

# SECTION 2

# DERIVATIONS

The procedure followed in deriving the spectrum and its subspectrum is as follows:

a. Derive a general expression for the autocorrelation of a phase-modulated wave. (This is a convenient way of obtaining the sub-carrier spectrum; however the general expression will not be employed to obtain the modulated carrier's spectrum, since less cumbersome means exist.)

b. Apply the above general expression to the specific case and thereby obtain the subspectrum.

c. Obtain the desired spectrum from the subspectrum by applying the following equation, which is derived in Appendix A.

$$y(t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \cdots \sum_{n_n=-\infty}^{\infty} \Big[ J_{n_1}(\beta_1) J_{n_2}(\beta_2) J_{n_3}(\beta_3) \cdots J_{n_n}(\beta_n) \sin(\omega_c + n_1\omega_1 + n_2\omega_2 + \cdots n_n\omega_n)\, t \Big]$$

This procedure is simplified by the fact that higher-order components can be neglected.

## 2.1 GENERAL EXPRESSION FOR AUTOCORRELATION FUNCTION OF PHASE MODULATED WAVE

Let

$$y(t) = \cos\left[\omega t + f(t)\right] , \qquad (1)$$

where f(t) represents a phase of the cosinusoidal (sub)carrier, which may assume any initial value since it is the change of phase with time that determines the resulting spectrum. Let the autocorrelation function of y(t) be denoted by $R_y(\tau)$ and the expected value of a quantity be denoted by $E\{\ \}$. By definition,

$$R_y(\tau) = E\left\{ y(t-\tau) \cdot y(t) \right\}, \tag{2}$$

$$R_y(\tau) = E\left\{\cos\left[\omega(t-\tau) + f(t-\tau)\right] \cdot \cos\left[\omega t + f(t)\right]\right\} \tag{3}$$

$$\begin{aligned} R_y(\tau) = \frac{1}{2} E\Big\{ &\cos\left[-\omega\tau + f(t-\tau) - f(t)\right] \\ &+ \cos\left[2\omega t - \omega\tau + f(t-\tau) + f(t)\right]\Big\} \end{aligned} \tag{4}$$

$$\begin{aligned} R_y(\tau) = \frac{1}{2} E\Big\{ &\cos(-\omega\tau) \cos\left[f(t-\tau) - f(t)\right] \\ &- \sin(-\omega\tau) \sin\left[f(t-\tau) - f(t)\right] \\ &+ \cos(-\omega\tau) \cos\left[2\omega t + f(t-\tau) + f(t)\right] \\ &- \sin(-\omega\tau) \sin\left[2\omega t + f(t-\tau) + f(t)\right]\Big\} \end{aligned} \tag{5}$$

$$\begin{aligned} R_y(\tau) = &\frac{1}{2} \cos\omega\tau \cdot E\left\{\cos\left[f(t-\tau) - f(t)\right]\right\} \\ &+ \frac{1}{2} \sin(\omega\tau) \cdot E\left\{\sin\left[f(t-\tau) - f(t)\right]\right\} \\ &+ \frac{1}{2} \cos(\omega\tau) \cdot E\left\{\cos\cancelto{0}{(2\omega t)} \cos\left[f(t-\tau) + f(t)\right]\right\} \\ &- \frac{1}{2} \cos(\omega\tau) \cdot E\left\{\sin\cancelto{0}{(2\omega t)} \sin\left[f(t-\tau) + f(t)\right]\right\} \\ &+ \frac{1}{2} \sin(\omega\tau) \cdot E\left\{\sin\cancelto{0}{(2\omega t)} \cos\left[f(t-\tau) + f(t)\right]\right\} \\ &+ \frac{1}{2} \sin(\omega\tau) \cdot E\left\{\cos\cancelto{0}{(2\omega t)} \sin\left[f(t-\tau) + f(t)\right]\right\} \end{aligned} \tag{6}$$

$$R_y(\tau) = \frac{1}{2}\left[\cos(\omega\tau)\, E\left\{\cos\left[f(t-\tau) - f(t)\right]\right\} + \frac{1}{2}\sin(\omega\tau)\, E\left\{\sin\left[f(t-\tau) - f(t)\right]\right\}\right] \tag{7}$$

Equation (7) is the general expression for the autocorrelation function of a phase modulated wave.

## 2.2 DETERMINATION OF SPECTRUM OF SUBCARRIER

Let the subcarrier whose spectrum we wish to determine be expressed as $y(t) = \cos \omega t + f(t)$ as explained in the preceding section. $y(t)$ and $f(t)$ are shown in Figs. 5 and 6, respectively.

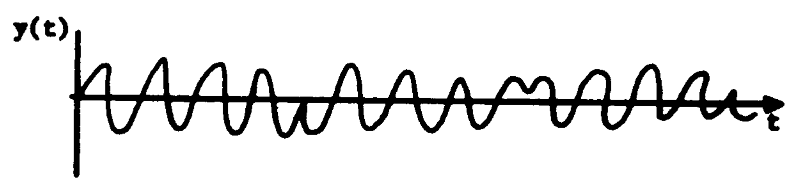



Fig. 5

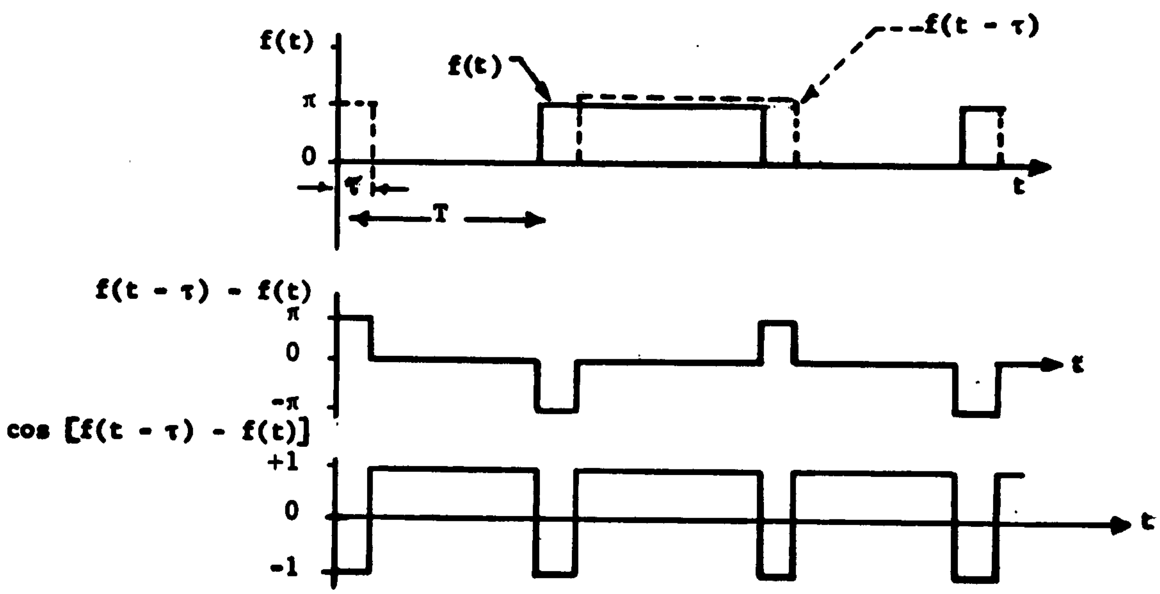



Fig. 6

Evaluating equation (7), we have:

$$E\left\{\cos\left[f(t-\tau)-f(t)\right]\right\} = \begin{bmatrix}\text{probability there is a phase}\\ \text{transition in time } \tau\end{bmatrix} \cdot$$

$$\left\{\begin{matrix}\cos\ f(t-\tau)-f(t)\\ \text{if there is a phase}\\ \text{transition}\end{matrix}\right\} + \begin{bmatrix}\text{probability there}\\ \text{is no phase tran-}\\ \text{sition in time } \tau\end{bmatrix} \cdot \left\{\begin{matrix}\cos\left[f(t-\tau)-f(t)\right]\\ \text{if there is no phase}\\ \text{transition}\end{matrix}\right\} \quad (8)$$

Similarly,

$$E\left\{\sin\left[f(t-\tau)-f(t)\right]\right\} = \begin{bmatrix}\text{probability there}\\ \text{is a phase tran-}\\ \text{sition in time } \tau\end{bmatrix} \cdot \left\{\begin{matrix}\sin\left[f(t-\tau)-f(t)\right]\\ \text{if there is a phase}\\ \text{transition.}\end{matrix}\right\}$$

$$+ \begin{bmatrix}\text{probability there is no}\\ \text{phase transition in time}\\ \tau\end{bmatrix} \cdot \begin{bmatrix}\sin\left[f(t-\tau)-f(t)\right]\\ \text{if there is no phase}\\ \text{transition in time } \tau\end{bmatrix} \quad (9)$$

Equations (8) and (9) become equations (10) and (11), respectively

$$\left[\frac{\tau}{T}\right]\left\{\cos(\pi)\right\} + \left[1-\frac{\tau}{T}\right]\left\{\cos 0\right\} = E\left\{\cos\left[f(t-\tau)-f(t)\right]\right\}$$

$$= -\frac{\tau}{T} + 1 - \frac{\tau}{T} = 1 - \frac{2\tau}{T} \quad \text{for } |\tau| \le T \quad (10)$$

$$E\left\{\sin\left[f(t-\tau)-f(t)\right]\right\} = \left[\frac{\tau}{T}\right]\left\{\cancel{\sin\pi}\right\}^{0} + \left[1-\frac{\tau}{T}\right]\left\{\cancel{\sin 0}\right\}^{0} = 0 \quad (11)$$

From equation (10) and Figs. 5 and 6, one can deduce that the plot of $E\left\{\cos\left[f(t-\tau)-f(t)\right]\right\}$ vs. the amount of time displacement, $\tau$, is as in Fig. 7.

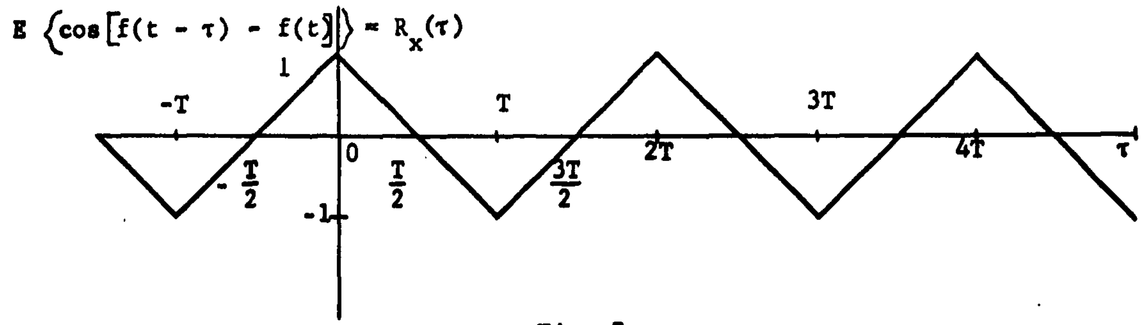



Fig. 7

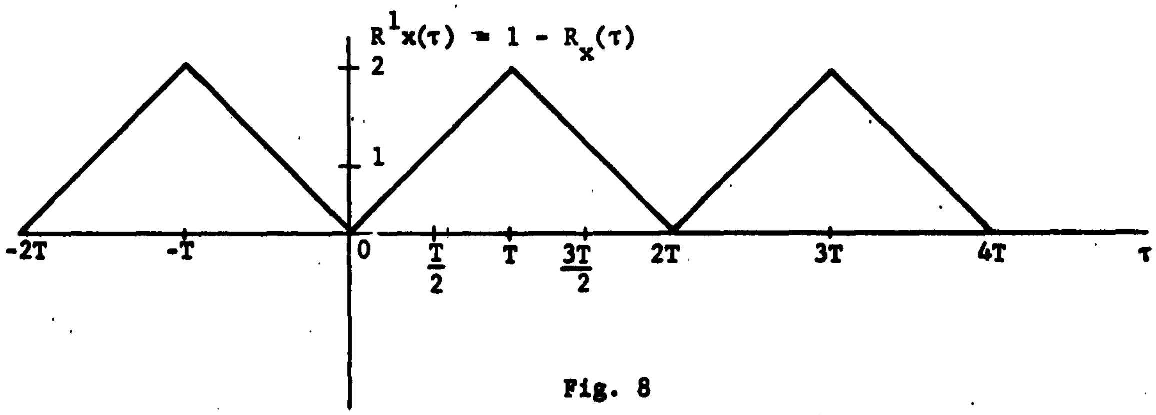



Fig. 8

Equation (7) then reduces to:

$$R_y(\tau) = \frac{1}{2} \cos (\omega\tau)\ E \left\{\cos \left[f(t - \tau) - f(t)\right]\right\} \tag{12}$$

$$= \frac{1}{2} \cos (\omega\tau)\ R_x(\tau)$$

The Fourier transform of equation (12) is the power spectral density of y(t), the subcarrier. That is,

$$\mathcal{F} \left[R_y(\tau)\right] = Sy(f) \tag{13}$$

$$Sy(f) = \mathcal{F} \left[\frac{1}{2} \cos (\omega\tau) \cdot E \left\{\cos \left[f(t - \tau) - f(t)\right]\right\}\right] \tag{14}$$

$$Sy(f) = \frac{1}{2} \mathcal{F} \left[\cos (\omega\tau)\right] * \mathcal{F} \left[E \left\{\cos \left[f(t - \tau) - f(t)\right]\right\}\right] \tag{15}$$

$$\frac{1}{2} \mathcal{F} \left[\cos (\omega\tau)\right] * \mathcal{F} \left[R_x(\tau)\right]$$

where the asterisk denotes convolution.

The function shown in Fig. 7, whose Fourier transform is to be taken as indicated in equation (15), can be expressed in terms of a Fourier series, the Fourier transform of which may be readily evaluated.

The Fourier series of the second term of the convolution of equation (15) is derived as follows:

The series can be expressed as

$$R_x(\tau) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos n\alpha\tau + b_n \sin n\alpha\tau) \qquad (16)$$

The "d-c" term is zero, as can be seen from Fig. 7; therefore, $a_o = 0$. Because $R_x(\tau)$ is an even function, $b_n = 0$ for all n.

It is easier to treat $R_x{}^1(\tau)$ as shown in Fig. 8 than $R_x(\tau)$ as shown in Fig. 7. Since $R_x{}^1(\tau) = - R_x(\tau) + 1$, and both $R_x(\tau)$ and $R_x{}^1(\tau)$ are even functions,

$$a_n \Big|_{R_x(\tau)} = -a_n \Big|_{R_x{}^1(\tau)} \qquad (17)$$

Simplifing the notation,

$$a_n = -a_n{}^1 \qquad (18)$$

$$a_n{}^1 = \frac{1}{T} \int_{-T}^{T} R_x{}^1(\tau) \cos \left[\frac{n\pi\tau}{T}\right] d\tau \qquad (19)$$

$$a_n{}^1 = \frac{1}{T} \int_{-T}^{o} \left[- \frac{2}{T}\right] \tau \cos\left[\frac{n\pi\tau}{T}\right] d\tau$$

$$+ \frac{1}{T} \int_{o}^{T} \left[\frac{2}{T}\right] \tau \cos\left[\frac{n\pi\tau}{T}\right] d\tau \qquad (20)$$

Reversing the limits of the first term of equation (20) would change the inherent sign, yielding equation (21).

$$a_n^{\,1} = \frac{4}{T^2}\left[\frac{\tau \sin\left[\frac{n\pi\tau}{T}\right]}{\left[\frac{n\pi}{T}\right]} + \frac{\cos\left[\frac{n\pi\tau}{T}\right]}{\left[\frac{n\pi}{T}\right]^2}\right]_0^T \tag{21}$$

$$a_n^{\,1} = \frac{4}{T^2}\left[\frac{\cos\,(n\pi)}{\left[\frac{n\pi}{T}\right]^2} - \frac{1}{\left[\frac{n\pi}{T}\right]^2}\right] \tag{22}$$

$$a_n^{\,1} = \frac{4}{n^2\pi^2}\left[\cos\,(n\pi) - 1\right] \tag{23}$$

using (18),

$$a_n = -\frac{4}{n^2\pi^2}\left[\cos\,(n\pi) - 1\right] \tag{24}$$

Therefore, the Fourier series is

$$R_x(\tau) = \frac{8}{\pi^2}\left[\cos\,\frac{\pi\tau}{T} + \frac{\cos\left[\frac{3\pi\tau}{T}\right]}{3^2} + \frac{\cos\left[\frac{5\pi\tau}{T}\right]}{5^2} + \ldots\right] \tag{25}$$

However, the ordinary way to write the angular frequency of a sinusoidal wave is in the form: $\cos\,(2\pi\ f\ t)$, where f is the angular frequency and its reciprocal is the period of the wave. Thus, if we let $f = 1/2T$

$$R_x(\tau) = \frac{8}{\pi^2}\left[\cos\,(2\pi\cdot f\cdot\tau) + \frac{\cos\,(2\pi\cdot 3f\cdot\tau)}{9} + \frac{\cos\,(2\pi\cdot 5f\cdot\tau)}{25} + \ldots\right] \tag{26}$$

Equation (26) shows that components exist at $\frac{1}{2T}$, $\frac{3}{2T}$, $\frac{5}{2T}$, etc; where T is the period. As with all periodic functions, the separation between successive components is $\frac{1}{T}$. Notice, too, that there are no even components (including the d-c component). We may now carry out the operation indicated in equation (15).

Let $\omega = 2\pi F$, where F is, as in equation (1), the angular frequency of the subcarrier.

Then substituting respective transforms into equation (15) and referring to equation (26).

$$Sy(f) = \frac{1}{2}\left[\frac{\delta(F) + \delta(-F)}{2}\right] * \frac{4}{\pi^2}\left[\delta(f) + \frac{1}{9}\ \delta(3f) + \frac{1}{25}\ (5f)\right.$$

$$\left. + \ldots\ldots + \delta(-f) + \frac{1}{9}\ \delta(-3f) + \frac{1}{25}\ \delta(-5f) + \ldots\right] \qquad (27)$$

where $\delta(x) = 1$ when $x = 0$

$\delta(x) = 0$ for $x \neq 0$

convolving,

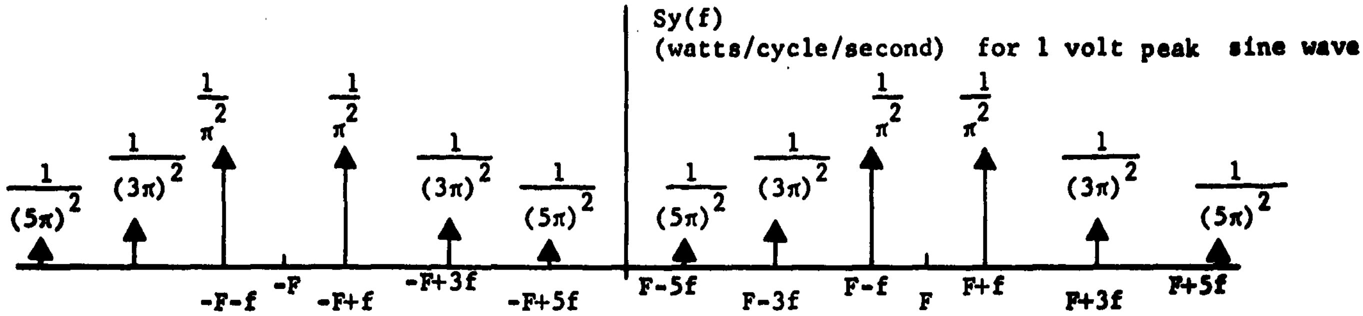
$$Sy(f) = \frac{1}{\pi^2}\left[\ldots\ldots\ \frac{1}{25}\ \delta(-F - 5f) + \frac{1}{9}\ \delta(-F - 3f) + \delta(-F - f)\right.$$

$$+ \delta(-F + f) + \frac{1}{9}\ \delta(-F + 3f) + \frac{1}{25}\ \delta(-F + 5f) + \ldots$$

$$\ldots + \frac{1}{25}\ \delta(F - 5f) + \delta(F - 3f) + \delta(F - f) + \ldots$$

$$\left.\delta(F + f) + \frac{1}{9}(F + 3f) + \frac{1}{25}\ (F + 5f) + \ldots\ldots\right] \qquad (28)$$

Equation (28) is the desired subspectrum and is plotted in Fig. 9.




$f = \frac{1}{2T}$ T = bit period 2T = wave period F = subcarrier frequency

Spectrum of Modulated Subcarrier
Fig. 9

The significance of the values of the delta functions of equation (28) plotted in Fig. 9 is that they represent the average power at each frequency. This is so, because they are the Fourier series of the auto-correlation function. If we had taken the Fourier Series of the function itself, we would have had to square the amplitudes and divide the result by two to obtain the power magnitudes.

The total power in the wave

$$y(t) = \frac{1}{T} \int_0^T y^2(t)\, dt = \frac{1}{2}\ .$$

The total power in the spectrum is, from Fig. 9

$$\sum_{n=-\infty}^{\infty} p_n = \frac{4}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right], \qquad (29)$$

but

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{1}{k^2}\right] = \frac{\pi^2}{8}\ ;$$

that is,

$$\lim_{N \to \infty}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \frac{1}{N^2}\right] = \frac{\pi^2}{8}$$

Thus,

$$\sum_{n=-\infty}^{\infty} p_n = \frac{4}{\pi^2}\left[\frac{\pi^2}{8}\right] = \frac{1}{2} ,$$

confirming our previous result.

## 2.3 DETERMINATION OF SPECTRUM OF MODULATED CARRIER

Following our third step outlined in the procedure, we determine the output of the phase modulator. It is specified that $\beta_{rms} = 1.5$ radians.

It is desired to determine the amount of phase each Fourier component of the subspectrum deviates the carrier, knowing that one volt peak amplitude cosine wave produces a peak deviation of the carrier of an amount:

$$\beta_{peak} = \sqrt{2}\ \ \beta_{rms}$$

$$\beta_{peak} = \sqrt{2}\ \ (1.5 \text{ radians})$$

$$\beta_{peak} = 2.12 \text{ radians / 1 volt}$$

For the $n^{th}$ Fourier Component if the peak amplitude is denoted by $V_n$ and the peak deviation by $\beta_n$ then

$$\beta_n = 2.12\ V_n \tag{30}$$

The numbers plotted in Fig. 9 are the two-sided powers associated with each Fourier component. Keeping this in mind, and in view of the fact that

$$p_n = \frac{V_n^2}{2} , \tag{31}$$

where $V_n$ denotes the peak amplitudes, we get Table 1 and

$$V_n = \sqrt{2p_n} = \sqrt{4p_{n1/2}}$$

$$V_n = 2\sqrt{p_{n1/2}} \qquad (32)$$

where $p_{n1/2}$ are the two sided power spectral densities.

TABLE 1

| f | (watts) $P_{n1/2}$ | $V_n$ (volt) | $\beta_n$(rad) | $J_o(\beta)$ | $J_1(\beta)$ |
|---|---|---|---|---|---|
| $F \pm f$ | $\frac{1}{\pi^2}$ | $\frac{2}{\pi}$ | 1.35 | 0.59 | 0.53 |
| $F \pm 3f$ | $\frac{1}{(3\pi)^2}$ | $\frac{2}{3\pi}$ | 0.44 | 0.95 | 0.21 |
| $F \pm 5f$ | $\frac{1}{(5\pi)^2}$ | $\frac{2}{5\pi}$ | 0.27 | 0.98 | 0.13 |

Let us consider six of the positive frequency components of the sub-spectrum shown in Fig. 9 to simultaneously phase-modulate a carrier. Then the resulting spectrum is as shown in Fig. 10, the values having been computed by applying equation (33):

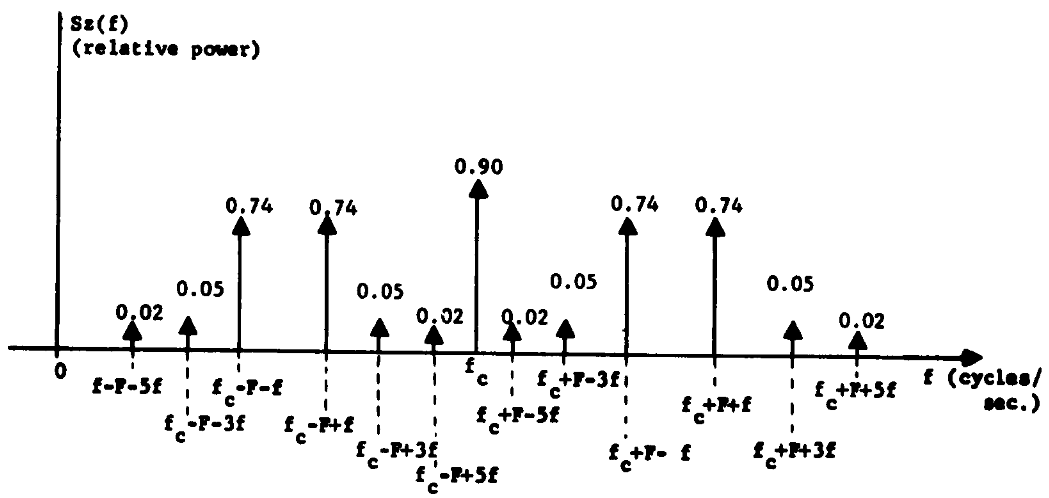
$$z(t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \cdots \sum_{n_n=-\infty}^{\infty}$$

$$\left\{ J_{n_1}(\beta_1) \cdot J_{n_2}(\beta_2) \cdot J_{n_3}(\beta_3) \cdots J_{n_n}(\beta_n) \left[ \sin (\omega_c + n_1\omega_1 + n_2\omega_2 \cdots n_n\omega_n)\, t \right] \right\} \qquad (33)$$

Evaluating the coefficients of equation (33),

| | | | Volt | Power |
|---|---|---|---|---|
| $f_c$ | : $(0.59)^2(0.95)^2(0.98)^2$ | $= (0.960)(0.903)(0.348)$ | = 0.302 | 0.90 |
| $f_c \pm (F+f)$ : | $(0.53)(0.59)\ (0.95)^2(0.98)^2$ | $= (0.53)(0.59)(0.96)(0.903)$ | = 0.271 | 0.74 |
| $f_c \pm (F+3f)$: | $(0.21)(0.95)(0.59)^2(0.98)^2$ | $= (0.21)(0.95)(0.348)(0.96)$ | = 0.067 | 0.05 |
| $f_c \pm (F+5f)$: | $(0.13)(0.98)(0.95)^2(0.59)^2$ | $= (0.13)(0.98)(0.903)(0.348)$ | = 0.040 | 0.02 |

The one-sided power spectrum (positive frequency, double side band) of the phase modulator output is then as in Fig. 10.




$f_c$ = carrier frequency
F = subcarrier frequency
f = 1/2 switching rate

Spectrum of Phase Modulator Output
Fig. 10

# APPENDIX A

## DERIVATION OF SPECTRUM FORMULA

Let the modulating signal, $f(t)$, be expressed as

$$f(t) = \sum_{i=1}^{n} a_i \sin \omega_i t \tag{A-1}$$

If $\omega_c$ is the carrier frequency, then the complex representation of the FM modulated signal, $z(t)$, is given by

$$z(t) = \exp\left\{j(\omega_c t + \sum_{i=1}^{n} a_i \sin \omega_i t)\right\}$$

$$= e^{j\omega_c t} \cdot e^{ja_1 \sin \omega_1 t} \cdots\cdots\cdots e^{ja_n \sin \omega_n t} \tag{A-2}$$

Now $e^{j\, a \sin \omega t}$ may be expressed in terms of Bessel functions as follows.

It is well known* that

$$e^{\frac{1}{2} x \left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \tag{A-3}$$

* Gray, Mathews and MacRobert, "A Treatise on Bessel Functions," 2nd edition, Macmillan & Co., London (1952), page 31.

Put $t = e^{j\theta}$. Then

$$\frac{1}{2} x \left(t - \frac{1}{t}\right) = \frac{1}{2} x \left(e^{j\theta} - e^{-j\theta}\right)$$

$$= jx \sin\theta$$

so that

$$e^{jx\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{jn\theta} \qquad \text{(A-4)}$$

With the aid of (A-4), one may rewrite (A-2) as

$$z(t) = e^{j\omega_c t} \cdot \sum_{n_1=-\infty}^{\infty} J_{n_1}(a_1)\, e^{jn_1\omega_1 t} \cdot \ldots\ldots.$$

$$\cdot \sum_{n_n=-\infty}^{\infty} J_{n_n}(a_n)\, e^{jn_n\omega_n t}$$

or, substituting $B_i$ for $a_i$,

$$z(t) = \sum_{n_1=-\infty}^{\infty} \ldots \sum_{n_n=-\infty}^{\infty} J_{n_1}(B_1) \ldots J_{n_n}(B_n)\, e^{j(\omega_c + n_1\omega_1 + \ldots n_n\omega_n)t} \qquad \text{(A-5)}$$

In a similar manner, it may be shown that if

$$z(t) = \sin\left(\omega_c t + \sum_{i=1}^{n} B_i \sin\omega_i t\right)$$

then

$$z(t) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_n=-\infty}^{\infty} J_{n_1}(B_1) \cdots J_{n_n}(B_n) \sin\left[(\omega_c + n_1\omega_1 + \cdots n_n\omega_n)t\right] \tag{A-6}$$

which is the formula used in the main part of the paper.

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